

# Action-angle variables for geodesic motions in Sasaki-Einstein spaces $Y^{p,q}$

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## Abstract

We use the action-angle variables to describe the geodesic motions in the 5-dimensional Sasaki-Einstein spaces  $Y^{p,q}$ . This formulation allows us to study thoroughly the complete integrability of the system. We find that the Hamiltonian involves a reduced number of action variables. Therefore one of the fundamental frequency is zero indicating a chaotic behavior when the system is perturbed.

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## 1 Introduction

There has been considerable interest recently in Sasaki-Einstein (SE) geometry [1]. In dimension five, an infinite family of explicit SE metrics  $Y^{p,q}$  on  $S^2 \times S^3$  has been constructed, where  $p$  and  $q$  are two coprime positive integers, with  $q < p$  [2].

A  $(2n - 1)$ -dimensional manifold  $M$  is a *contact manifold* if there exists a 1-form  $\eta$  (called a contact 1-form) on  $M$  such that

$$\eta \wedge (d\eta)^{n-1} \neq 0. \quad (1)$$

The *Reeb vector field*  $\xi$  dual to  $\eta$  satisfies:

$$\eta(\xi) = 1 \quad \text{and} \quad \xi \lrcorner d\eta = 0, \quad (2)$$

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where  $\lrcorner$  is the operator dual to the wedge product.

A contact Riemannian manifold  $(Y_{2n-1}, g_{Y_{2n-1}})$  is Sasakian if its metric cone  $C(Y_{2n-1}) = Y_{2n-1} \times \mathbb{R}_+$  with the metric

$$ds^2(C(Y_{2n-1})) = dr^2 + r^2 ds^2(Y_{2n-1}), \quad (3)$$

is Kähler [3]. Here  $r \in (0, \infty)$  may be considered as a coordinate on the positive real line  $\mathbb{R}_+$ . If the Sasakian manifold is Einstein, the metric cone is Ricci-flat and Kähler, i.e. Calabi-Yau.

The orbits of the Reeb vector field  $\xi$  may or may not close. If the orbits of the Reeb vector field  $\xi$  are all closed, then  $\xi$  integrates to an isometric  $U(1)$  action on  $(Y_{2n-1}, g_{Y_{2n-1}})$ . Since  $\xi$  is nowhere zero this action is locally free. If the  $U(1)$  action is in fact free, the Sasakian structure is said to be *regular*. Otherwise it is said to be *quasi-regular*. If the orbits of  $\xi$  are not all closed, the Sasakian structure is said to be *irregular* and the closure of the 1-parameter subgroup of the isometry group of  $(Y_{2n-1}, g_{Y_{2n-1}})$  is isomorphic to a torus  $\mathbb{T}^n$  [1].

The homogeneous SE metric on  $S^2 \times S^3$ , known as  $T^{1,1}$ , represents an example of regular Sasakian structure with  $SU(2) \times SU(2) \times U(1)$  isometry. The  $Y^{p,q}$  spaces have isometry  $SU(2) \times U(1) \times U(1)$  and for  $4p^2 - 3q^2$  a square they are examples of quasi-regular SE manifolds. The geometries  $Y^{p,q}$  with  $4p^2 - 3q^2$  not a square are irregular SE spaces.

In a recent paper [4] the constants of motion for geodesic motions in the five-dimensional spaces  $Y^{p,q}$  have been explicitly constructed. This task was achieved using the complete set of Killing vectors and Killing-Yano tensors of these toric SE spaces. A multitude of constants of motion have been generated, but only five of them are functionally independent implying the complete integrability of geodesic flow on  $Y^{p,q}$  spaces.

The complete integrability of geodesics permits us to construct explicitly the action-angle variables. The formulation of an integrable system in these variables represents a useful tool for developing perturbation theory. The action-angle variables define an  $n$ -dimensional surface which is a topological torus (Kolmogorov-Arnold-Moser (KAM) tori) [5].

Our motivation for studying the action-angle parametrization of the phase space for geodesic motions in SE spaces comes from recent studies of non-integrability and chaotic behavior of some classical configuration of strings in the context of AdS/CFT correspondence. It was shown that certain classical string configurations in  $AdS_5 \times T^{1,1}$  [6] or  $AdS_5 \times Y^{p,q}$  [7] are chaotic. There were used numerical simulations or an analytic approach through the Kovacic's algorithm [8].

The purpose of this paper is to describe the geodesic motions in the SE spaces  $Y^{p,q}$  in the action-angle formulation. We find that the Hamiltonian (energy) involves only four action variables which have the corresponding frequencies different of zero. One of the fundamental frequency is zero foreshadowing a chaotic behavior when the system is perturbed.

The paper is organized as follows. In the next Section we give the necessary preliminaries regarding the metric and the constants of motion for geodesics on

$Y^{p,q}$  spaces. In Sec. 3 we perform the separation of variables and give an action-angle parametrization of the phase space. The paper ends with conclusions in Sec. 4.

## 2 $Y^{p,q}$ spaces

The AdS/CFT correspondence represents an important advancement in string theory. A large class of examples consists of type *IIB* string theory on the background  $AdS_5 \times Y_5$  with  $Y_5$  a 5-dimensional SE space. In the frame of AdS/CFT correspondence  $Y^{p,q}$  spaces have played a central role as they provide an infinite class of dualities.

We write the metric of the 5-dimensional  $Y^{p,q}$  spaces [2, 9, 10] as

$$ds_{Y^{p,q}}^2 = \frac{1-cy}{6}(d\theta^2 + \sin^2 \theta d\phi^2) + \frac{1}{w(y)q(y)}dy^2 + \frac{q(y)}{9}(d\psi - \cos \theta d\phi)^2 + w(y) \left[ d\alpha + \frac{ac - 2y + cy^2}{6(a-y^2)}(d\psi - \cos \theta d\phi) \right]^2, \quad (4)$$

where

$$w(y) = \frac{2(a-y^2)}{1-cy}, \quad q(y) = \frac{a-3y^2+2cy^3}{a-y^2}. \quad (5)$$

This metric is Einstein with  $\text{Ric } g_{Y^{p,q}} = 4g_{Y^{p,q}}$  for all values of the constants  $a, c$ . For  $c = 0$  the metric takes the local form of the standard homogeneous metric on  $T^{1,1}$  [11]. Otherwise the constant  $c$  can be rescaled by a diffeomorphism and in what follows we assume  $c = 1$ .

A detailed analysis of the SE metric  $Y^{p,q}$  [2] showed that for  $0 \leq \theta \leq \pi$  and  $0 \leq \phi \leq 2\pi$  the first two terms of (4) give the metric on a round two-sphere. The two-dimensional  $(y, \psi)$ -space defined by fixing  $\theta$  and  $\phi$  is fibred over this two-sphere. The range of  $y$  is fixed so that  $1-y > 0$ ,  $a-y^2 > 0$  which implies  $w(y) > 0$ . Also it is demanded that  $q(y) \geq 0$  and that  $y$  lies between two zeros of  $q(y)$ , i.e.  $y_1 \leq y \leq y_2$  with  $q(y_i) = 0$ . To be more specific, the roots  $y_i$  of the cubic equation

$$a - 3y^2 + 2y^3 = 0, \quad (6)$$

are real, one negative ( $y_1$ ) and two positive, the smallest being  $y_2$ . All of these conditions are satisfied if the range of  $a$  is

$$0 < a < 1. \quad (7)$$

Taking  $\psi$  to be periodic with period  $2\pi$ , the  $(y, \psi)$ -fibre at fixed  $\theta$  and  $\phi$  is topologically a two-sphere. Finally, the period of  $\alpha$  is chosen so as to describe a principal  $S^1$  bundle over  $B_4 = S^2 \times S^2$ . For any  $p$  and  $q$  coprime, the space  $Y^{p,q}$  is topologically  $S^2 \times S^3$  and one may take [11, 2]

$$0 \leq \alpha \leq 2\pi\ell, \quad (8)$$

where

$$\ell = \frac{q}{3q^2 - 2p^2 + p(4p^2 - 3q^2)^{1/2}}. \quad (9)$$

To put the formulas in a simpler forms, in that follows we introduce also

$$f(y) = \frac{a - 2y + y^2}{6(a - y^2)}, \quad (10)$$

$$p(y) = \frac{w(y)q(y)}{6} = \frac{a - 3y^2 + 2y^3}{3(1 - y)}. \quad (11)$$

The conjugate momenta to the coordinates  $(\theta, \phi, y, \alpha, \psi)$  are:

$$\begin{aligned} P_\theta &= \frac{1-y}{6} \dot{\theta}, \\ P_y &= \frac{1}{6p(y)} \dot{y}, \\ P_\alpha &= w(y) \left( \dot{\alpha} + f(y) (\dot{\psi} - \cos \theta \dot{\phi}) \right), \\ P_\psi &= w(y) f(y) \dot{\alpha} + \left[ \frac{q(y)}{9} + w(y) f^2(y) \right] (\dot{\psi} - \cos \theta \dot{\phi}), \\ P_\phi &= \frac{1-y}{6} \sin^2 \theta \dot{\phi} - \cos \theta P_\psi \\ &= \frac{1-y}{6} \sin^2 \theta \dot{\phi} - \cos \theta w(y) f(y) \dot{\alpha} - \cos \theta \left[ \frac{q(y)}{9} + w(y) f^2(y) \right] \dot{\psi} \\ &\quad + \cos^2 \theta \left[ \frac{q(y)}{9} + w(y) f^2(y) \right] \dot{\phi}, \end{aligned} \quad (12)$$

with overdot denoting proper time derivative.

The Hamiltonian describing the motion of a free particle is

$$H = \frac{1}{2} g^{\mu\nu} P_\mu P_\nu, \quad (13)$$

which for the  $Y^{p,q}$  metric (4) and using the momenta (12) has the form:

$$\begin{aligned} H &= \frac{1}{2} \left\{ 6p(y) P_y^2 + \frac{6}{1-y} \left( P_\theta^2 + \frac{1}{\sin^2 \theta} (P_\phi + \cos \theta P_\psi)^2 \right) + \frac{1-y}{2(a-y^2)} P_\alpha^2 \right. \\ &\quad \left. + \frac{9(a-y^2)}{a-3y^2+2y^3} \left( P_\psi - \frac{a-2y+y^2}{6(a-y^2)} P_\alpha \right)^2 \right\}. \end{aligned} \quad (14)$$

Starting with the complete set of Killing vectors and Killing-Yano tensors of the SE spaces  $Y^{p,q}$  it is possible to find quite a lot of integrals of motions [4, 12, 13]. However the number of functionally independent constants of motion is only five implying the complete integrability of geodesic flow on  $Y^{p,q}$  spaces. For example we can choose as independent conserved quantities the energy

$$E = H, \quad (15)$$

the momenta corresponding to the cyclic coordinates  $(\phi, \psi, \alpha)$

$$\begin{aligned} P_\phi &= c_\phi, \\ P_\psi &= c_\psi, \\ P_\alpha &= c_\alpha, \end{aligned} \tag{16}$$

where  $(c_\phi, c_\psi, c_\alpha)$  are some constants, and the total  $SU(2)$  angular momentum

$$\vec{J}^2 = P_\theta^2 + \frac{1}{\sin^2 \theta} (P_\phi + \cos \theta P_\psi)^2 + P_\psi^2. \tag{17}$$

### 3 Action-angle variables

The connection between completely integrable systems and toric geometry in the symplectic setting is described by the classical Liouville-Arnold theorem [5, 14]. A dynamical system defined by a given Hamiltonian  $H$  on a  $2n$ -dimensional symplectic manifold  $(M^{2n}, \omega)$  is called Liouville integrable if it admits  $n$  functionally independent first integrals in involution. In other words, there are  $n$  functions  $\mathbf{F} = (f_1 = H, f_2, \dots, f_n)$  such that  $df_1 \wedge \dots \wedge df_n \neq 0$  almost everywhere and

$$\{f_i, f_j\} = 0, \quad \forall i, j. \tag{18}$$

Let  $\mathbf{F}_c = (f_1 = E, f_2 = c_2, \dots, f_n = c_n)$  by a common invariant level set. If  $\mathbf{F}_c$  is regular, compact and connected, then it is diffeomorphic to the  $n$ -dimensional Lagrangian torus. For  $n$  degrees of freedom the motion is confined to an  $n$ -torus

$$\mathbb{T}^n = \underbrace{S^1 \times S^1 \times \dots \times S^1}_{n \text{ times}}. \tag{19}$$

These are called *invariant tori* and never intersects taking into account the uniqueness of the solution to the dynamical system, expressed as a set of coupled ordinary differential equations.

In a neighborhood of  $\mathbf{F}_c$  there are action-angle variables  $\mathbf{J}, \mathbf{w} \bmod 2\pi$ , such that the symplectic form becomes

$$\omega = \sum_{i=1}^n dJ_i \wedge dw_i, \tag{20}$$

and the Hamiltonian  $H$  depends only on actions  $J_1, \dots, J_n$ . An action variable  $J_i$  specifies a particular  $n$ -torus  $\mathbb{T}^n$  and is constant since the tori are invariant. The location on the torus is specified by  $n$  angle variables  $w_i$ . Even the system is integrable, the dynamics on the singular set (where the differentials of the integrals  $f_1, \dots, f_n$  are dependent) can be quite complicated [15].

In the case of the geodesic motions on  $Y^{p,q}$ , for the beginning, we fix a level surface  $\mathbf{F} = (H, P_\phi, P_\psi, P_\alpha, \vec{J}^2) = \mathbf{c}$  of the mutually commuting constants of motion (15)–(17). The differentials of the chosen first integrals are real analytic [4]. Then it suffices to require their functional independence at least at one point

[16] to apply the Liouville-Arnold theorem. Further we introduce the generating function for the canonical transformation from the coordinates  $(\mathbf{p}, \mathbf{q})$ , where  $\mathbf{p}$  are the conjugate momenta (12) to the coordinates  $\mathbf{q} = (\theta, \phi, y, \alpha, \psi)$ , to the action-angle variables  $(\mathbf{J}, \mathbf{w})$  as the indefinite integral

$$S(\mathbf{q}, \mathbf{c}) = \int_{\mathbf{F}=\mathbf{c}} \mathbf{p} \cdot d\mathbf{q}. \quad (21)$$

Since the Hamiltonian (14) has no explicit time dependence, we can write

$$S(\mathbf{q}, \mathbf{c}) = W(\mathbf{q}, \mathbf{c}) - Et, \quad (22)$$

with the Hamilton's characteristic function

$$W = \sum_i \int p_i dq_i. \quad (23)$$

In the case of geodesic motions in SE spaces  $Y^{p,q}$  the variables in the Hamilton-Jacobi equation are separable and consequently we seek a solution of the Hamilton's characteristic function of the form

$$W(y, \theta, \phi, \psi, \alpha) = W_y(y) + W_\theta(\theta) + W_\phi(\phi) + W_\psi(\psi) + W_\alpha(\alpha). \quad (24)$$

The *action variables*  $\mathbf{J}$  are defined as integrals over complete period of the orbit in the  $(p_i, q_i)$  plane

$$J_i = \oint p_i dq_i = \oint \frac{\partial W_i(q_i; c)}{\partial q_i} dq_i \quad (\text{no summation}). \quad (25)$$

$J_i$ 's form  $n$  independent functions of  $c_i$ 's and can be taken as a set of new constant momenta.

Conjugate *angle variables*  $w_i$  are defined by the equations:

$$w_i = \frac{\partial W}{\partial J_i} = \sum_{j=1}^n \frac{\partial W_j(q_j; J_1, \dots, J_n)}{\partial J_i} \quad (26)$$

having a linear evolution in time

$$w_i = \omega_i t + \beta_i \quad (27)$$

with  $\beta_i$  other constants of integration and  $\omega_i$  are frequencies associated with the periodic motion of  $q_i$ .

Hamilton characteristic functions associated with cyclic variables are

$$\begin{aligned} W_\phi &= P_\phi \phi = c_\phi \phi, \\ W_\psi &= P_\psi \psi = c_\psi \psi, \\ W_\alpha &= P_\alpha \alpha = c_\alpha \alpha, \end{aligned} \quad (28)$$

where  $c_\phi, c_\psi, c_\alpha$  are the constants introduced in (16).

The corresponding action variables are

$$\begin{aligned} J_\phi &= 2\pi c_\phi, \\ J_\psi &= 2\pi c_\psi, \\ J_\alpha &= 2\pi \ell c_\alpha. \end{aligned} \quad (29)$$

Taking into account (14) and (23), the Hamilton-Jacobi equation becomes

$$\begin{aligned} E = \frac{1}{2} \left\{ 6p(y) \left( \frac{\partial W_y}{\partial y} \right)^2 + \frac{6}{1-y} \left[ \left( \frac{\partial W_\theta}{\partial \theta} \right)^2 + \frac{1}{\sin^2 \theta} (c_\phi + \cos \theta c_\psi)^2 \right] \right. \\ \left. + \frac{1-y}{2(a-y^2)} c_\alpha^2 + \frac{9(a-y^2)}{a-3y^2+2y^3} \left[ c_\psi - \frac{a-2y+y^2}{6(a-y^2)} c_\alpha \right]^2 \right\}. \end{aligned} \quad (30)$$

This equation can be written as follows

$$\begin{aligned} & \left( \frac{\partial W_\theta}{\partial \theta} \right)^2 + \frac{1}{\sin^2 \theta} (c_\phi + \cos \theta c_\psi)^2 \\ &= \frac{1-y}{3} E - p(y)(1-y) \left( \frac{\partial W_y}{\partial y} \right)^2 - \frac{(1-y)^2}{12(a-y^2)} c_\alpha^2 \\ & \quad - \frac{3(a-y^2)(1-y)}{2(a-3y^2+2y^3)} \left[ c_\psi - \frac{a-2y+y^2}{6(a-y^2)} c_\alpha \right]^2. \end{aligned} \quad (31)$$

We observe that the LHS of this equation depends only  $\theta$  and independent of  $y$ . Therefore we may set

$$\left( \frac{\partial W_\theta}{\partial \theta} \right)^2 + \frac{1}{\sin^2 \theta} (c_\phi + \cos \theta c_\psi)^2 = c_\theta^2, \quad (32)$$

with  $c_\theta$  another constant. From the last equation we can evaluate the action variable

$$J_\theta = \oint d\theta \sqrt{c_\theta^2 - \frac{(c_\phi + c_\psi \cos \theta)^2}{\sin^2 \theta}}. \quad (33)$$

The limits of integrations are defined by the roots  $\theta_-$  and  $\theta_+$  of the expressions in the square root sign and a complete cycle of  $\theta$  involves going from  $\theta_-$  to  $\theta_+$  and back to  $\theta_-$ .

This integral can be evaluated by elementary means or using the complex integration method of residues which turns out to be more efficient [14, 17]. For the evaluation of the integral (33) we put  $\cos \theta = t$ , extend  $t$  to a complex variable  $z$  and interpret the integral as a closed contour integral in the complex  $z$ -plane. Consider the integrand in (33)

$$\frac{\sqrt{-(c_\theta^2 + c_\psi^2)z^2 - 2c_\phi c_\psi z + c_\theta^2 - c_\psi^2}}{z^2 - 1} = \frac{\sqrt{-(c_\theta^2 + c_\psi^2)}}{z^2 - 1} \sqrt{(z - t_+)(z - t_-)}, \quad (34)$$

where the roots

$$t_{\pm} = \frac{-c_{\phi}c_{\psi} \pm c_{\theta}\sqrt{c_{\theta}^2 + c_{\psi}^2 - c_{\phi}^2}}{c_{\theta}^2 + c_{\psi}^2}, \quad (35)$$

are the turning points of the  $t$ -motion. They are real for

$$c_{\theta}^2 + c_{\psi}^2 - c_{\phi}^2 \geq 0, \quad (36)$$

and situated in the interval  $(-1, +1)$ .

For  $z > t_+$  we specify the right side of the square root from (34) as positive. We cut the complex  $z$ -plane from  $t_-$  to  $t_+$  and the closed contour integral of the integrand is a loop enclosing the cut in a clockwise sense. The contour can be deformed to a large circular contour plus two contour integrals about the poles at  $z = \pm 1$ . After simple evaluation of the residues and the contribution of the large contour integral we finally get:

$$J_{\theta} = 2\pi \left[ \sqrt{c_{\theta}^2 + c_{\psi}^2} - c_{\phi} \right]. \quad (37)$$

For the action variable corresponding to  $y$  coordinate we have from (31)

$$\begin{aligned} \frac{\partial W_y}{\partial y} = & \left\{ \frac{1-y}{a-3y^2+2y^3} E - \frac{3}{a-3y^2+2y^3} c_{\theta}^2 \right. \\ & - \frac{9(a-y^2)(1-y)}{2(a-3y^2+2y^3)^2} c_{\psi}^2 + \frac{3(a-2y+y^2)(1-y)}{2(a-3y^2+2y^3)^2} c_{\psi}c_{\alpha} \\ & \left. - \frac{(1-y)(2a+a^2-6ay-2y^2+2ay^2+6y^3-3y^4)}{8(a-3y^2+2y^3)^2(a-y^2)} c_{\alpha}^2 \right\}^{\frac{1}{2}}. \end{aligned} \quad (38)$$

It is harder to evaluate the action variable  $J_y$  in a closed analytic form taking into account the complicated expression (38). In fact the closed-form of  $J_y$  is not at all illuminating. More important is the fact that  $J_y$  depends only of four constants of motion:  $E, J_{\theta}, J_{\alpha}, J_{\psi}$ . In consequence the energy depends only on four action variables  $J_y, J_{\theta}, J_{\alpha}, J_{\psi}$  representing a reduction of the number of action variables entering the expression of the energy of the system.

For the angular variable  $w_{\phi}$  we have

$$w_{\phi} = \frac{1}{2\pi} J_{\phi} + \frac{\partial W_{\theta}}{\partial J_{\phi}}. \quad (39)$$

Putting  $\cos \theta = t$  the second term is

$$\begin{aligned} \frac{\partial W_{\theta}}{\partial J_{\phi}} = & -\frac{1}{2\pi} \int dt \frac{(J_{\phi} + J_{\theta})t^2 + J_{\psi}t}{(1-t^2)\sqrt{-(J_{\phi} + J_{\theta})^2t^2 - 2J_{\phi}J_{\psi}t + (J_{\theta}^2 + 2J_{\theta}J_{\phi} - J_{\psi}^2)}} \\ = & \frac{1}{2\pi} \int \frac{dt}{1-t^2} \frac{\mathfrak{d}t^2 + \mathfrak{e}t}{\sqrt{\mathfrak{a} + \mathfrak{b}t + \mathfrak{c}t^2}}, \end{aligned} \quad (40)$$



where

$$\begin{aligned}
\mathfrak{a} &= J_\theta^2 + 2J_\theta J_\phi - J_\psi^2, \\
\mathfrak{b} &= -2J_\theta J_\psi, \\
\mathfrak{c} &= -(J_\theta + J_\phi)^2, \\
\mathfrak{d} &= J_\theta + J_\phi, \\
\mathfrak{e} &= J_\psi.
\end{aligned} \tag{41}$$

We necessitate the following integrals [18]:

$$\begin{aligned}
I_1(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}; t) &= \int \frac{dt}{\sqrt{\mathfrak{a} + \mathfrak{b}t + \mathfrak{c}t^2}} \\
&= \frac{-1}{\sqrt{-\mathfrak{c}}} \arcsin\left(\frac{2\mathfrak{c}t + \mathfrak{b}}{\sqrt{-\Delta}}\right)
\end{aligned} \tag{42}$$

evaluated for  $\mathfrak{c} < 0$ ,  $\Delta = 4\mathfrak{a}\mathfrak{c} - \mathfrak{b}^2 < 0$ , and

$$\begin{aligned}
I_2(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}; t) &+ \int \frac{dt}{t\sqrt{\mathfrak{a} + \mathfrak{b}t + \mathfrak{c}t^2}} \\
&= \frac{1}{\sqrt{-\mathfrak{a}}} \arctan\left(\frac{2\mathfrak{a} + \mathfrak{b}t}{2\sqrt{-\mathfrak{a}}\sqrt{\mathfrak{a} + \mathfrak{b}t + \mathfrak{c}t^2}}\right)
\end{aligned} \tag{43}$$

evaluated for  $\mathfrak{a} < 0$ . That is the case of the constants  $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$  taking into account the constraint (36).

Using these integrals we get for the angular variable  $w_\phi$

$$\begin{aligned}
w_\phi &= \frac{1}{2\pi} J_\phi - \frac{\mathfrak{d}}{2\pi} I_1(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}; \cos \theta) \\
&\quad - \frac{\mathfrak{d} + \mathfrak{e}}{4\pi} I_2(\mathfrak{a} + \mathfrak{b} + \mathfrak{c}, \mathfrak{b} + 2\mathfrak{c}, \mathfrak{c}; \cos \theta - 1) \\
&\quad - \frac{\mathfrak{e} - \mathfrak{d}}{4\pi} I_2(\mathfrak{a} - \mathfrak{b} + \mathfrak{c}, \mathfrak{b} - 2\mathfrak{c}, \mathfrak{c}; \cos \theta + 1).
\end{aligned} \tag{44}$$

The explicit evaluation of the angular variables  $w_\theta, w_\psi, w_\alpha, w_y$  is again intricate due to the absence of a simple closed-form for the action variable  $J_y$ . However, it is remarkable the fact that one of the fundamental frequencies (27)

$$\omega_i = \frac{\partial H}{\partial J_i}, \tag{45}$$

is zero, namely

$$\omega_\phi = \frac{\partial H}{\partial J_\phi} = 0, \tag{46}$$

since the action  $J_\phi$  does not enter the expression of the energy.

The topological nature of the flow of each invariant torus (19) depends on the properties of the frequencies  $\omega_i$  (27). There are essentially two cases [19]:

1. The frequencies  $\omega_i$  are nonresonant

$$k_i \omega_i \neq 0 \quad \text{for all} \quad 0 \neq k_i \in \mathbb{Z}^n. \quad (47)$$

Then, on this torus each orbit is dense and the flow is ergodic.

2. The frequencies  $\omega_i$  are resonant or rational dependent

$$k_i \omega_i = 0 \quad \text{for some} \quad 0 \neq k_i \in \mathbb{Z}^n. \quad (48)$$

The prototype is  $\omega = (\omega_1, \dots, \omega_{n-m}, 0, \dots, 0)$  with  $1 \leq m \leq n-1$  zero frequencies and  $(\omega_1, \dots, \omega_{n-m})$  nonresonant frequencies.

The KAM theorem [5] describes how an integrable system reacts to small non-integrable deformations. The KAM theorem states that for nearly integrable systems, i.e. integrable systems plus sufficiently small conservative Hamiltonian perturbations, most tori survive, but suffer a small deformation. However the resonant tori which have rational ratios of frequencies get destroyed and motion on them becomes chaotic.

In the case of geodesics on  $Y^{p,q}$  space, the frequencies are resonant (46) giving way to chaotic behavior when the system is perturbed. The analysis performed in [7] confirms the present results produced in the action-angle approach.

## 4 Conclusions

The action-angle formulation for  $Y^{p,q}$  spaces gives us a better understanding of the dynamics of the geodesic motions in these spaces. In spite of the complexity of the evaluation of some variables, we are able to prove that the energy of the system depends on a reduced number of action variables signaling a degeneracy of the system.

This fact corroborates a similar result obtained in the case of geodesic motions in the homogeneous SE space  $T^{1,1}$  [17]. The metric on  $T^{1,1}$  may be written by utilizing the fact that it is a  $U(1)$  bundle over  $S^2 \times S^2$ . The evaluations of all action and angle variables was completely done putting them in closed analytic forms. In the case of the space  $T^{1,1}$  the isometry is  $SU(2)^2 \times U(1)$  and there are two pairs of fundamental frequencies which are resonant. The degeneracy of these two pairs of frequencies may be removed by a canonical transformation to new action-angle variables. Finally the Hamiltonian governing the motions on  $T^{1,1}$  can be written in terms of only *three* action variables for which the corresponding frequencies are different from zero.

In conclusion, the action-angle approach offers a strong support for the assertion that certain classical string configurations in  $AdS_5 \times Y_5$  with  $Y_5$  in a large class of Einstein spaces is non-integrable [20, 21]. It would be interesting to extend the action-angle formulations to other five-dimensional SE spaces as well as to their higher dimensional generalizations relevant for the predictions of the AdS/CFT correspondence.

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